Model Linearization and $H_\infty$ Controller Design for a Quadrotor Unmanned Air Vehicle: Simulation Study

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Abstract—This paper describes the architecture of a quadrotor and its linearized dynamic model. The attitude of the quadrotor in the model state-space variables can be calculated with rotational matrix formed from quaternions instead of finite small angular displacement approximation. Based on $H_\infty$ continuous time control method, a state feedback controller is designed to control the translational position and heading of the quadrotor.

Keywords—Closed-loop feedback control, $H_\infty$ controller.

I. INTRODUCTION

Motion control of Unmanned Aerial Vehicles (UAVs) is a challenging problem and has to be implemented through rigorous calculations of system modelling and tuning of such models to replicate the actual system. Various researches have done attitude and altitude control on quadrotor UAVs [1][10]. With attitude control, it is just a basis for translational position. However, heading control of a quadrotor UAV will be more useful especially for hovering applications such as precise unloading, aerial photography, area mapping and monitoring. The dynamics model of a quadrotor consists of coupling and nonlinearity. To minimize the coupling effect, a standard quadrotor is designed to have four independent propellers attached at each corner of the quadrotor in a symmetric manner. A pair of propellers will be rotating in counterclockwise direction (for standard propeller) while the other pair in the reverse direction to produce zero net yaw [9]. The problem of nonlinearity has also been addressed by using several methods such as adaptive control algorithm [12], sliding mode variable structure control algorithm [10], backstepping design algorithm [2], and feedback linearization algorithm [3].

In this paper, a feedback linearization algorithm is implemented. The reasons are as follows:

- Feedback linearization algorithm does not require the exact model of predicted system which is difficult to be obtained but needed by adaptive control algorithm to meet local differential need [11].

- Sliding mode variable structure control algorithm has the disadvantage of chattering phenomena [11] in crossing both sides of the sliding mode surfaces and this may lead to vibration and mechanical wear and tear issues.

- The backstepping algorithm has the “explosion of complexity” problem caused by the repeated differentiations of virtual controllers [10].

With the feedback linearized model, a LQR controller which is a subset of the $H_\infty$ controller [4] is designed which can provide a phase margin of more than 60 degree and gain margin of more than 0.5 [13]. Moreover, instead of PID controller [3], $H_\infty$ controller is chosen because it can save the try and error tuning time.

The contents of this paper are organized as follows: In Section II, the mathematical linearized model of a quadrotor is described. In Section III, a continuous $H_\infty$ controller is designed and the simulation results of the controller are in Section IV. Lastly, conclusion and future work are found in Section V.

II. MATHEMATICAL MODELLING

A. Dynamics of a Quadrotor

A quadrotor is actuated by four motor driven propellers with each provides lift thrust, $T$, and drag torque, $Q$, [1][2].

$$T = C_T \rho_\text{air} A r^2 \Omega^2 = c'_T \Omega^2 \quad (1)$$

$$Q = -\text{sgn}(\Omega)C_Q \rho_\text{air} A r^2 \Omega^2 = -\text{sgn}(\Omega) c'_Q \Omega^2 \quad (2)$$

where $C_T$ and $C_Q$ are non-dimensional thrust and torque coefficients, $\rho_{\text{air}}$ is air density, $A$ is propeller disk area, $r$ is the propeller disk radius and $\Omega$ is motor angular speed.

The dynamics of a quadrotor are analyzed in different frames with the Cartesian x-y-z axes follow the right-hand-rule. Fig. 1 shows a quadrotor with four fixed-angle-rotors where sum of the lift thrusts is in $z$-axis of body frame $\{B\}$ while sum of the drag torques are able to cancel out each other. Fig. 2 shows arbitrary attitude of a quadrotor in the view of Earth
sgn is not bound to the origin of frame) from frame \{E\} to origin of \{B\} as in Fig. 3. The origin of \{B\} and \{C\} are the same point. The \(z\)-axis of \{C\} is in the same direction of the \(z\)-axis of \{E\} such that \{C\} is formed from \{E\} after translation from origin of \{E\} to origin of \{B\} followed by rotation about \(z\)-axis by angle \(\theta_z\). \{B\} is formed from \{C\} after rotation about \(\hat{k}\)-axis by angle \(\theta_y\). Both unit vectors \(\hat{h}\) and \(\hat{k}\) are parallel to the horizontal \(xy\)-plane of \{C\} and \{E\}. \(\hat{h}\), \(\hat{k}\) and \(z_C\) are perpendicular to each other. Hence, \(\hat{h}\), \(z_B\) and \(z_C\) are on the same plane.

Solving with Euler’s axis-angle rotation matrix, the following results can be obtained:

\[
\begin{align*}
\theta_z &= \arctan2(r_{21} - r_{12}, r_{11} + r_{22}) \\
\theta_y &= \arccos(r_{33}), \theta_y \in [0, \pi] \\
\phi &= \arctan2(-r_{32}, -r_{31})
\end{align*}
\]

where \(r_{ij}\) are the elements of rotational matrix \(E_{\text{g}}R\) in (3) and \(\arctan2(y, x)\) computes \(\arctan(y/x)\) and uses signs of both \(x\) and \(y\) to determine the quadrant in which the angle lies.

To avoid the loss of generality of the dynamic model of the quadrotor, the following assumptions are made [3]:

- The quadrotor is a symmetrical rigid body;
- The origin of inertial, geometric center and the centroid of the quadrotor are in the same position of the origin \{B\};
- The air density, gravity, propeller thrust and torque coefficients are constants during the flight;
- Coriolis effect on angular velocity of the quadrotor is negligible.

For quadrotor, it has been assumed to be symmetrical in physical quantities and geometrical structure and therefore the inertia matrix in body frame, \(J\), will be a diagonal matrix.

\[
J = \begin{bmatrix}
J_{xx} & 0 & 0 \\
0 & J_{yy} & 0 \\
0 & 0 & J_{zz}
\end{bmatrix}
\]

B. Linearized System

The system equation after linearization at \(\theta_{xy0} = 0\) and hence \(T_0 = mg/4\) will be

\[
\dot{X} = AX + BU
\]

where

\[
U = \begin{bmatrix}
\frac{\partial T_x}{\partial \theta_x} \\
\frac{\partial T_x}{\partial \theta_y} \\
\frac{\partial T_x}{\partial \phi}
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
\dot{\theta}_x \\
\dot{\theta}_y \\
\dot{\phi}
\end{bmatrix}
\]

\[
\begin{align*}
\dot{\theta}_x &= c_{\theta_x} \dot{c}_v + c_{\theta_x} c_{\omega_x} \dot{c}_\omega - s_{\theta_x} \dot{c}_v \\
\dot{\theta}_y &= c_{\theta_y} \dot{c}_v + c_{\theta_y} c_{\omega_y} \dot{c}_\omega - s_{\theta_y} \dot{c}_v \\
\dot{\phi} &= c_{\phi} \dot{c}_v + c_{\phi} c_{\omega_z} \dot{c}_\omega - s_{\phi} \dot{c}_v
\end{align*}
\]
\[ A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \Delta \alpha = \left\| \bar{C}_c R - \bar{C}_c R_0 \right\|_{\text{max}} B \delta T = 2B \alpha \delta T \]

The error of linear accelerations due to linearization at \( \theta_{xy} = 0 \),

\[ \Delta \alpha_c = \begin{bmatrix} 0 & 0 & \frac{l}{J_z} & -\frac{l}{J_z} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \delta T \]

The controlled outputs are the translational position and heading (yaw angle) of the quadrotor,

\[ s_0 = s_{\text{ref}} \]

\[ \theta_{z0} = \theta_{z\text{ref}} \]

which can be substituted into (7) and (8).

To implement the state feedback, control [4], the following assumptions are made:

- All of the state-space variables in (6) are assumed to be measurable with sensors or derivable from algorithms such as the horizontal \( x, y \) positions can be obtained by using GPS or SLAM algorithms [5], or height position, \( z \), can be measured by ultrasonic sensor or barometer, translational velocities can be obtained from optical flow [6][7], rotational velocities can be obtained with an inertial measurement unit (IMU) and rotational angles can be obtained from (3), (4) and (9) with the rotational matrix in (3) can be calculated from quaternions updated by INS algorithm [8],

\[ R = \begin{bmatrix} 2(w^2 + \chi^2) - 1 & 2\chi y - wz & 2\chi z + wy \\ 2\chi y + wz & 2\chi^2 - y^2 - 1 & 2\chi z - wy \\ 2\chi z - wy & 2\chi z + wy & 2(w^2 + z^2) - 1 \end{bmatrix} \]

- The input disturbances and measurement or algorithm errors of state-space variables are small and negligible.

With these assumptions, the quadrotor system become,

\[ \dot{X} = AX + BU + EW \]

\[ \Sigma : \begin{bmatrix} Y \\ Z \end{bmatrix} \begin{bmatrix} X \\ C_z X + D_z \end{bmatrix} \]

where \( W \) only consists of wind gust input disturbance while measurement or algorithm errors are negligible. The controlled output, \( Z \), consists of normalized state-space variables which...
associate with reference inputs, errors due to linearization at $\theta_{xy}=0$ and normalized control units. Hence,

$$
E = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
Z = \begin{bmatrix}
\delta^x \theta_x \\
\delta^y \theta_y \\
\delta^x s_x \\
\delta^y s_y \\
\delta^x z \\
\delta^y z \\
\delta^x \alpha_x \\
\delta^y \alpha_y \\
\delta^x \nu_x \\
\delta^y \nu_y
\end{bmatrix},
$$

where a normalized variable, $\tilde{X}$,

$$
\tilde{X} = \frac{X}{X_{\max}}
$$

(13)

The maximum values of state-space variables which associate with reference inputs should be chosen based on the overall system performance requirements and tolerance of error in measurement or algorithm. On the other hand, the maximum value of control input should be chosen based on the hardware limit and the efficiency of energy control.

From (10), (11) and (13),

$$
C_x = \begin{bmatrix}
\frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\partial \theta_{\max}}
\end{bmatrix}
$$

(14)

For the system in (12), $(A, B)$ is controllable, $D_2$ is of maximal column rank and $(A, B, C_x, D_y)$ has no invariant zeros on the imaginary axis. Hence the system is a regular case [4].

The following algebraic Riccati equation (H$\infty$-ARE) is to be solved,

$$
A^T P + PA + C_x^T C_x + P E E^T P/\gamma^2 - (PB + C_x^T D_2)(D_2^T D_2)^{-1}(D_2^T C_x + B^T P) = 0
$$

for a unique positive semi-definite solution $P \geq 0$ and $\gamma > \gamma_*$ to reduce the control effort. The feedback gain matrix,

$$
F = -(D_2^T D_2)^{-1}(D_2^T C_x + B^T P)
$$

and the $H_\infty$-$\gamma$-suboptimal state feedback law,

$$
U = F \tilde{X}
$$

where

$$
\tilde{X} = \begin{bmatrix}
\text{sgn}(\delta \theta_x) \min \| \delta \theta_x \|_{\theta_{\max}} \\
\text{sgn}(\delta s_x) \min \| \delta s_x \|_{s_{\max}} \\
\text{sgn}(\delta z) \min \| \delta z \|_{z_{\max}} \\
\text{sgn}(\delta \alpha_x) \min \| \delta \alpha_x \|_{\alpha_{\max}} \\
\text{sgn}(\delta \nu_x) \min \| \delta \nu_x \|_{\nu_{\max}}
\end{bmatrix}
$$

such that the magnitude of the state-space variables which associates with the reference inputs is capped at each designated maximum value in (13) to avoid over-reaction of the controller and followed by large deviation from the linearized model.
IV. SIMULATION RESULTS

The parameters of a quadrotor in Table I are used in simulation. Table II shows the chosen parameter values which are used to calculate the feedback controller gain, $F$, in the feedback control law in (14),

$$F = \begin{bmatrix}
-0.4814 & -0.4814 & 0.4814 & 0.4814 \\
0.8221 & -0.8221 & 0 & 0 \\
0 & 0 & 0.7831 & -0.7831 \\
-2.0639 & -2.0639 & -2.0639 & -2.0639 \\
0.8259 & -0.8259 & 0 & 0 \\
0 & 0 & 0.7868 & -0.7868 \\
0 & 0 & -3.5414 & 3.5414 \\
3.7174 & -3.7174 & 0 & 0 \\
-1.3180 & -1.3180 & -1.3180 & -1.3180 \\
0 & 0 & -0.6893 & 0.6893 \\
0.7236 & -0.7236 & 0 & 0 \\
-0.2133 & -0.2133 & 0.2133 & 0.2133
\end{bmatrix}$$

In order to make the simulation results close to the reality, the following statements are implemented during the simulation process:

- Nonlinear model of the quadrotor with neglected Coriolis Effect on small angle changes is used as system plant.
- Gaussian noise with zero mean, standard deviation of 1/30 and maximum magnitude of 0.1 is mixed with each state-space variable before the control law in (14), to represent the measurement noise and algorithm error.
- Wind gust in the simulation,

$$\delta W = \begin{bmatrix}
\cos \left( \frac{2\pi}{40} (t - 70) \right) \\
\cos \left( \frac{2\pi}{40} (t - 70) \right) \\
\cos \left( \frac{2\pi}{40} (t - 70) \right) \\
0 & 0 & 0 \\
\cos \left( \frac{2\pi}{40} (t - 70) \right)
\end{bmatrix}, \forall \delta \leq 80$$

- The quadrotor is assumed to have half throttle during hovering and hence the thrust force input to the system is ranged between 0 and $2T_o$ ($\delta T$ is capped at $T_o$).
- The controller is running at 20Hz or control cycle of 50ms with zero order hold on control units to simulate a discrete time controller.
- The initial states are all zeroes as the quadrotor is resting on a platform before taking off.
- The references in the simulation,

$$\begin{bmatrix}
\delta \theta_{ref} \\
\delta \phi_{ref} \\
\delta \psi_{ref} \\
\delta \theta_{ref}
\end{bmatrix} = \begin{bmatrix}
-\pi \\
10 \\
-10 \\
1
\end{bmatrix}, \forall t > 0$$

Fig. 4 shows state-space variables associated with the reference inputs against time. It is clear that the designed controller is able to bring the quadrotor to the reference positions even with the presence of the simulated wind gust together with small measurement noise and algorithm error on feedback states.

$$\begin{array}{c|c|c}
\text{Parameter} & \text{Value} & \text{Unit} \\
\hline
g & 9.81 & \text{ms}^{-2} \\
T_o & 4.128 & \text{N} \\
\delta \theta_{max} & 0.049 & \text{rad} \\
\delta \phi_{max} & 0.05 & \text{m} \\
\delta \psi_{max} & 0.05 & \text{m} \\
\gamma & 17.9 & \\
\hline
\end{array}$$

where $T_o = mg/4$ and assume the quadrotor requires half throttle to hover.

Fig. 5 shows the path taken by the quadrotor and Fig. 6 shows the attitude of quadrotor against time. During the existence of wind gust ($60 \leq t \leq 80$), the controller tilted the quadrotor against the wind gust while holding on the reference altitude to avoid the quadrotor from crashing.
reference set points with the existence of artificial wind gust together with small measurement noise and algorithm error.

B. Future Work

In the future, the physical experimental studies will be conducted such that the linearized dynamic model together with the $H_\infty$ control method will be implemented on a real quadrotor for performance analysis.

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